Discrete Mathematics: Combinatorics and Graph Theory

Exam 3 Solution

Instructions. Solve any 5 questions and state which 5 you would like graded. Write neatly and show your work to receive full credit. You must sign the attendance sheet when returning your booklet. Good luck!

1. Solve the following recurrence relations.

   (a) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$
   $x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2 \Rightarrow a_n = \alpha_1(2)^n + \alpha_2 n(-2)^n$. When $n = 0 : 6 = \alpha_1$. When $n = 1 : 8 = -6(2) + 2\alpha_2 \Rightarrow \alpha_2 = -2$. Therefore $a_n = 6(2)^n - 2n(2)^n$.

   (b) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.
   $x^2 - 2x^2 - 5x + 6 = 0$. By the rational root test $x = \pm 1, \pm 2, \pm 3, \pm 6$. We find that $x = 1$ is a root so that $(x-1)(x^2 - x - 6) \Rightarrow (x-1)(x-3)(x+2) = 0 \Rightarrow x = 1, x = 3, x = -2$. The general solution is then $a_n = \alpha_1(1)^n + \alpha_2(3)^n + \alpha_3(-2)^n$. Applying the initial conditions produces the system of equations:

   $$\begin{aligned} 7 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3 \\ -4 &= a_1 = \alpha_1 + 3\alpha_2 - 2\alpha_3 \\ 8 &= a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3 \end{aligned}$$

   Solving the system gives $\alpha_1 = 5, \alpha_2 = -1$ and $\alpha_3 = 3$. The solution is thus $a_n = 5 - 3^n + 3(-2)^n$.

   (c) $a_n = na_{n-1}$, $a_1 = 1$
   We can recognize the recurrence relation as the factorial function $a_n = n!$

   (d) $a_n = a_{n-1} + 2n$ with $a_0 = 2$
   When $n = 1$, $a_1 = a_0 + 2$. Iterating hints at $a_n = n^2 + n + 2 = n(n+1) + 2$. Base case: $a_1 = a_0 + 2 \times 1 = 2$. IH: $a_k = k(k+1) + 2$ for some k. Inductive step: $a_{k+1} = (k+1)^2 + (k+1) + 2 = k^2 + 2k + 1 + k + 1 + 2 = k^2 + k + 1 + 2k + 1 + 2 = k(k+1) + 2 + 2(k+1) = a_k + 2(k+1)$.

   (e) $a_{n+2} + a_{n+1} - 6a_n = 2^n$ for $n \geq 0$.
   $x^2 + x - 6 = 0 \Rightarrow (x-2)(x+3) = 0 \Rightarrow x = 2, x = -3$. Therefore $a_n^{(h)} = \alpha_1(2)^n + \alpha_2(-3)^n$. Given the nonhomogeneous part is exponential, we would try $C2^n$, however $x = 2$ is one of the roots of the characteristic polynomial. Try $Cn2^n$ and substitute: $C(n+2)2^{n+2} + C(n+1)2^{n+1} - 6Cn2^n = 2^n$. Simplifying we find $10C2^n = 2^n \Rightarrow C = 1/10 \Rightarrow a_n^{(p)} = (n/10)2^n$. The final solution is $a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1(2)^n + \alpha_2(-3)^n + (n/10)2^n$.

2. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively.

   $$P(E) = 1 - P(\text{all boys}) - P(\text{all girls}) = 1 - 2 \times (0.5)^n$$
   $$p(F) = P(\text{no boys}) + P(\text{one boy}) = (0.5)^n + \binom{n}{1}(0.5)^n = (n+1)(0.5)^n$$
   $$P(E, F) = P(\text{one boy}) = \binom{n}{1}(0.5)^n$$

   Are E and F independent if

   (a) $n = 2$.
   $$P(E, F) = 0.5 \neq 0.375 = P(E)P(F)$$

   (b) $n = 4$.
   $$P(E, F) = 0.25 \neq 0.2734 = P(E)P(F)$$

(c) $n = 5$.

$$P(E, F) = 0.15625 \neq 0.1757 = P(E)P(F)$$

3. Consider a password generated by selecting characters from a three letter alphabet α, β or γ which must use each letter at least once. How many such passwords of length 8 are there?
   Let X be the set of passwords that do not contain α, Y be the set of passwords that do not contain β and Z be the set of passwords that do not contain C. The question asks for $|X \cup Y \cup Z|$. There are 2^8 passwords that just contain β and γ. By the same reasoning $|X| = |Y| = |Z| = 2^8$. Passwords that do not contain α or β just contain γ. There is only one such password ($\gamma\gamma\gamma\gamma\gamma\gamma\gamma\gamma$). By this reasoning $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$. Passwords that do not contain α, β, γ do not exist therefore $|\alpha \cap \beta \cap \gamma| = 0$. By the inclusion-exclusion principle $|X \cup Y \cup Z| = 3 \times 2^8 - 3 \times 1 + 0 = 3 \times 2^8 - 3$. To find the answer to the original question, subtract the number above from the total number of passwords, which is 3^8. Therefore $3^8 - (3 \times 2^8 - 3)$.

4. Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. Suppose you randomly choose a machine and put in a quarter. If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time? If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?
   $P(S) = 0.5$, $P(J|S) = 0.1$, $P(J|\bar{S}) = 0.2$. We want to find $P(\bar{S}|\bar{J})$ and $P(\bar{S}|J)$.

$$\begin{aligned} P(\bar{S}|\bar{J}) &= \frac{P(\bar{J}|\bar{S})P(\bar{S})}{\bar{J}} \\ &= \frac{P(\bar{J}|\bar{S})P(\bar{S})}{P(\bar{J}|\bar{S})P(\bar{S}) + P(\bar{J}|S)P(S)} \\ &= \frac{0.8 \times 0.5}{0.8 \times 0.5 + 0.9 \times 0.5} \\ &= 0.471 \end{aligned}$$

   Similarly

$$\begin{aligned} P(\bar{S}|J) &= \frac{P(J|\bar{S})P(\bar{S})}{P(J|\bar{S})P(\bar{S}) + P(J|S)P(S)} \\ &= \frac{0.2 \times 0.5}{0.2 \times 0.5 + 0.1 \times 0.5} \\ &= 0.667 \end{aligned}$$

5. Derive a formula for the k^{th} factorial moment of the Poisson distribution. Evaluate the expression

$$\mathbb{E}[X(X-1)(X-2)\cdots(X-k+1)]$$

   Hint: write the expression as a summation and simplify completely.

$$\begin{aligned} \mathbb{E}[X(X-1)(X-2)\cdots(X-k+1)] &= \sum_{x=0}^{\infty} x(x-1)\cdots(x-k+1)e^{-\lambda}\frac{\lambda^x}{x!} \\ &= \lambda^k \sum_{x=k}^{\infty} e^{-\lambda}\frac{\lambda^{x-k}}{(x-k)!} \\ &= \lambda^k \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} \\ &= \lambda^k \end{aligned}$$

6. Define X_1 and X_2 as independent Poisson random variables with parameters λ_1 and λ_2. Let $\lambda = \lambda_1 + \lambda_2$ and define $Z = X_1 + X_2$. Find the distribution of Z by evaluating the expression $P(Z = z)$.
Note that if $X_1 + X_2 = Z$ then $X_2 = Z - X_1$.

$$\begin{aligned}
P(Z = z) &= \sum_{j=0}^{z} P(X_1 = j, X_2 = z - j) \\
&= \sum_{j=0}^{z} P(X_1 = j) P(X_2 = z - j) \\
&= \sum_{j=0}^{z} \frac{e^{-\lambda_1} \lambda_1^j}{j!} \frac{e^{-\lambda_2} \lambda_2^{z-j}}{(z-j)!} \\
&= \sum_{j=0}^{z} \frac{1}{j!(z-j)!} e^{-\lambda_1} \lambda_1^j e^{-\lambda_2} \lambda_2^{z-j} \\
&= \sum_{j=0}^{z} \frac{z!}{j!(z-j)!} \frac{e^{-\lambda_1} \lambda_1^j e^{-\lambda_2} \lambda_2^{z-j}}{z!} \\
&= \frac{e^{-\lambda}}{z!} \sum_{j=0}^{z} \binom{z}{j} \lambda_1^j \lambda_2^{z-j} \\
&= \frac{e^{-\lambda}}{z!} (\lambda_1 + \lambda_2)^z \\
&= \frac{e^{-\lambda}}{z!} \lambda^z
\end{aligned}$$

7. Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n. Solve this recurrence relation to find a formula for d_n.
When $n = 1$, $|A_1| = 2$ and so $d_1 = 2$. When $n = 2$, we have $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and so $|A_2| = 4 - 1 = 3$. When $n = 3$ we have $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ where $|A_3| = 2(2 \times 2 - 1 \times 1) - 1(1 \times 2 - 1 \times 0) + 0(1 \times 1 - 2 \times 0) = 4$.
In general, expanding along the top row gives 2 times the determinant of the matrix obtained by deleting the first row and first column minus the determinant of the matrix obtained by deleting the first row and second column. The first matrix is A_{n-1} with determinant d_{n-1}. The second of the smaller matrices has just one nonzero entry in its first column. Expanding the determinant along the first column produces d_{n-2}. We can conjecture that the recurrence relation is $d_n = 2d_{n-1} - d_{n-2}$. Considering the first few terms $d_1 = 2, d_2 = 3, d_3 = 4, \cdots$ we can guess that $d_n = n + 1$. Plugging in we find that $n + 1 = 2n - (n - 1)$.